CHARACTERIZING BU BY HOMOTOPY GROUPS

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In this note we provide proofs for two facts related to connected topological K-theory, expected by Ib Madsen.

Let BU denote the connected K-theory spectrum, and let $\operatorname{Im} J$ denote the connected image of J spectrum. For an odd prime p we prove that $BU_p^\wedge$ is characterized by its homotopy groups and v_1-periodic structure. Thereafter we prove that spectrum maps from $BU_p^\wedge$ to $B\operatorname{Im} J_p^\wedge$ are detected on homotopy groups.

The purpose for proving these Lemmas characterizing properties of BU by homotopy group data, is to complete the calculation of M. Bökstedt, W. C. Hsiang and I. Madsen's cyclotomic trace invariant $TC(\mathbb{Z}_p^\wedge, p)$ receiving a map from $K(\mathbb{Z}_p^\wedge)$, for odd p. See Corollary 3, whose proof is due to Bökstedt and Madsen. Their proof of its hypothesis will appear in [2].

Lemma 1. *Suppose X is a p-complete spectrum, with $\pi_* X \cong \pi_* BU_p^\wedge$ abstractly, and*

$$\pi_*(X; \mathbb{F}_p) \cong \pi_*(BU; \mathbb{F}_p)$$

as $\mathbb{F}_p[v_1]$-modules. Then $X \simeq BU_p^\wedge$.

Let S denote the sphere spectrum, and $H\mathbb{F}_p$ the mod p Eilenberg–MacLane spectrum. For any spectrum Y, let Y/p denote the cofiber of the "multiplication by p" map. In all proofs, let all spectra be implicitly completed at p.

Proof. We first prove that $X/p \simeq BU/p$. The map $\Sigma^{2p-2} S/p \xrightarrow{v_1} S/p$ smashed with X induces the v_1-module structure on mod p homotopy, and appears in the bottom line of :

$$\begin{tikzcd}
\Sigma^{2p-2} BU/p \arrow[r, "v_1"] & BU/p \arrow[r] \arrow[d, dashed, "f"] & \vee_{i=1}^{p-1} \Sigma^{2i} H\mathbb{F}_p \arrow[d, "\simeq"] & \\
\Sigma^{2p-2} X/p \arrow[r, "v_1"] & X/p \arrow[r] & C(v_1) \arrow[r] & \Sigma^{2p-1} X/p
\end{tikzcd}$$

where the horizontal lines are (co–)fibration sequences (cf. [1] Corollary 4.8). By the hypothesis about the $\mathbb{F}_p[v_1]$-module structure on $\pi_*(X/p)$, the mapping cofiber $C(v_1)$ has the same homotopy groups as $\vee_{i=1}^{p-1} \Sigma^{2i} H\mathbb{F}_p$. The k-invariants of its Postnikov tower thus sit in degrees 3 through $2p-3$ in the mod p Steenrod algebra, where all the groups vanish. Hence $C(v_1)$ is a wedge of Eilenberg–MacLane spectra, and we can choose an equivalence as in the diagram above.

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Consider the composite $BU/p \to \vee_i \Sigma^{2i} H\mathbb{F}_p \to C(v_1) \to \Sigma^{2p-1}X/p$. By the usual Atiyah–Hirzebruch spectral sequence argument (all groups sit in odd degrees), this map must induce a null homotopic map of underlying spaces.

For the corresponding spectrum level conclusion, recall that the spectrum BU is the connected cover of the mapping telescope for the canonical degree two map (multiply by the Bott class) on the suspension spectrum of its zero'th space. Also note that there is an equivalence between maps from a periodic spectrum or its connected cover to a connected periodic spectrum (cf. [4] Proposition II.3.8).

Similarly, the spectrum BU/p can be recovered as the connected cover of the mapping telescope for the v_1-map on the suspension spectrum of its zeroth space. Hence by a Milnor exact sequence argument there are no essential spectrum maps from BU/p to $\Sigma^{2p-1}X/p$. (The $\lim^1$-term vanishes by completeness.)

Thus there exists a lifting f making the diagram above commute. On homotopy, f induces a homomorphism of isomorphic $\mathbb{F}_p[v_1]$-modules which is an isomorphism modulo the ideal generated by v_1. Hence $\pi_*(f)$ is an isomorphism and f is an equivalence.

Next consider the diagram :

$$\begin{tikzcd}
BU \arrow[r, "p\cdot"] & BU \arrow[r, "q"] \arrow[d, dashed, "h"] & BU/p \arrow[r] \arrow[d, "f"] & \Sigma BU \\
X \arrow[r, "p\cdot"] & X \arrow[r] & X/p \arrow[r] & \Sigma X
\end{tikzcd}$$

By an Atiyah–Hirzebruch spectral sequence argument as above, there are no essential maps $BU \to \Sigma X$. Hence the composite $f \circ q$ lifts to a map $h\colon BU \to X$. This is a map between p-complete spectra with abstractly isomorphic homotopy groups, which induces an equivalence with mod p-coefficients, and is hence an equivalence. □

I thank Marcel Bökstedt for noting the need for the spectrum level extensions in the above argument.

Let q be a topological generator for the p-adic units $\mathbb{Z}_p^\times \subset \mathbb{Z}_p^\wedge$. Extend the Adams operation ψ^q to a spectrum map $BU_p^\wedge \to BU_p^\wedge$ mapping as $q^{-n}\psi^q$ on the $2n$th space, and let $\operatorname{Im} J_p^\wedge$ be the fiber of $\psi^q - 1\colon BU_p^\wedge \to BU_p^\wedge$.

Lemma 2. *The spectrum maps from $BU_p^\wedge$ to $B\operatorname{Im} J_p^\wedge$ are precisely the p-adic multiples of the cofiber map corresponding to $\psi^q - 1$. They are all detected on homotopy groups.*

Proof. Consider the diagram :

$$\begin{tikzcd}
 & BU \arrow[d, dashed, "\phi"'] \arrow[dr] & & \\
BU \arrow[r, "\psi^q - 1"] & BU \arrow[r] & B\operatorname{Im} J \arrow[r] & BBU
\end{tikzcd}$$

Again, there are no essential maps from BU to BBU, so any spectrum map $BU \to B\operatorname{Im} J$ lifts to a spectrum map $\phi\colon BU \to BU$. This is a reduced K-theory operation, and by [3] Formula 2.3 and Remark 2.10, ϕ is expressible as a series :

$$\phi = \sum_{r\geq 0}\sum_{i\geq 0} a_{r,i}(\psi^{p^r+i} - \psi^i)$$

with $a_{r,i} \in \mathbb{Z}_p^\wedge$, where $a_{r,i} = 0$ if $i \geq p^{r+1} - p^r$ or $p|i$, and the sum may be infinite in r. Note that $\psi^{p^r+i} - \psi^i$ may be expressed as $(\psi^{p^r+i} - 1) - (\psi^i - 1)$ with both $p^r + i$ and i prime to p in the cases where $a_{r,i} \neq 0$, except when $r = i = 0$.

We claim that any operation $\psi^n - 1$ with n a p-adic unit factors through $\psi^q - 1$. This follows for $n = q^i$ with i a natural number from the factorization $\psi^{q^i} - 1 = (\psi^q - 1)(\psi^{q^{i-1}} + \cdots + \psi^q + 1)$, and for negative i using the operation ψ^{-q}. As q is a topological generator, the general case follows by completion.

On the other hand, any operation ϕ factoring through $\psi^q - 1$ has $a_{0,0} = 0$. For if it were nonzero, let $j = v_p(a_{0,0}) < \infty$ be the p-adic valuation, let $i = (p-1)p^j$, and consider homotopy groups in degree $2i$. For any p-adic unit n, $v_p(n^i - 1) \geq j + 1$, so

$$v_p(\pi_{2i}(\phi)) = v_p(a_{0,0}) = j$$

which precludes ϕ from mapping into the image of $\psi^q - 1$, as $v_p(\pi_{2i}(\psi^q - 1)) = v_p(q^i - 1) \geq j + 1$ by the above (q is a p-adic unit).

Hence the subspace of operations ϕ which factor through $\psi^q - 1$ is contained in the closed ideal where $a_{0,0} = 0$ in the compact Hausdorff ring of all stable operations, and contains the dense subset of such operations which are finite sums of Adams operations. As the former subspace is a continuous image of the whole compact ring, it is itself compact, thus closed, and is therefore the whole ideal.

In conclusion, the coefficient $a_{0,0}$ precisely classifies the spectrum maps $BU \to B\operatorname{Im} J$, as we wanted to prove. □

Let J denote the connective image of J spectrum, i.e. the connective cover of the fiber of $\psi^q - 1\colon K \to K$, or equivalently the connective cover of the K-localization of S, where K is the periodic K-theory spectrum.

Corollary 3. *(Bökstedt–Madsen) Suppose there is a map of p-complete spectra $J_p^\wedge \vee BJ_p^\wedge \to TC(\mathbb{Z}_p^\wedge, p)$ with cofiber X, injective on homotopy groups, such that $\pi_* X \cong \pi_* SU_p^\wedge$ abstractly, and $\pi_*(X; \mathbb{F}_p) \cong \pi_*(SU; \mathbb{F}_p)$ as $\mathbb{F}_p[v_1]$-modules. Then*

$$TC(\mathbb{Z}_p^\wedge, p) \simeq J_p^\wedge \vee BJ_p^\wedge \vee SU_p^\wedge .$$

Proof. By suspending Lemma 1 once, we find $X \simeq SU$. Thus there is a fibration sequence of spectra

$$J \vee BJ \longrightarrow TC(\mathbb{Z}_p^\wedge, p) \longrightarrow SU \xrightarrow{\Sigma e} BJ \vee BBJ$$

Desuspending the map Σe once, we are led to consider $e\colon BU \to J \vee BJ$, which is assumed zero on homotopy groups. There are no essential maps $BU \to J$ which are zero on homotopy groups, due to the corresponding fact for maps $BU \to BU$. Any map $BU \to BJ$ which is zero on homotopy groups factors through the 1-connected cover $B\operatorname{Im} J$, so by Lemma 2 the map e (and thus Σe) is null homotopic, from which the splitting follows (lift the identity map of SU). □

References

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